

Semi-regular biorthogonal pairs and generalized Riesz bases

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Abstract

In this paper we define the notion of semi-regular biorthogonal pairs what is a generalization of regular biorthogonal pairs in Ref. [2] and show that if $(\{\phi_n\}, \{\psi_n\})$ is a semi-regular biorthogonal pair, then $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases. This result improves the results of Ref. [1, 2, 3] in the regular case.

1 Introduction

Let \mathcal{H} be a Hilbert space with inner product $(\cdot|\cdot)$, $\mathbf{e} = \{e_n\}$ an ONB in \mathcal{H} and $\{\phi_n\}$ a sequence in \mathcal{H} . In Ref. [2], the author has defined an operator $T_{\mathbf{e}}$ on $D_{\mathbf{e}} \equiv \text{Span}\{e_n\}$ by

$$T_{\mathbf{e}} \left(\sum_{k=0}^n \alpha_k e_k \right) = \sum_{k=0}^n \alpha_k \phi_k.$$

By using this operator $T_{\mathbf{e}}$, the author has investigated the relationship between a regular biorthogonal pair $(\{\phi_n\}, \{\psi_n\})$ and the notions of Riesz bases and semi-Riesz bases. Here, $(\{\phi_n\}, \{\psi_n\})$ is a pair of Riesz bases if there exists an ONB $\mathbf{e} = \{e_n\}$ in \mathcal{H} such that both $T_{\mathbf{e}}$ and $T_{\mathbf{e}}^{-1}$ are bounded, and $(\{\phi_n\}, \{\psi_n\})$ is a pair of semi-Riesz bases if there exists an ONB $\mathbf{e} = \{e_n\}$ in \mathcal{H} such that either $T_{\mathbf{e}}$ or $T_{\mathbf{e}}^{-1}$ are bounded. In this paper we consider the following operators in \mathcal{H} defined by a sequence $\{\phi_n\}$ in \mathcal{H} and an ONB $\mathbf{e} = \{e_n\}$ in \mathcal{H} :

$$\begin{aligned} T_{\phi, \mathbf{e}} &\equiv \sum_{k=0}^{\infty} \phi_k \otimes \bar{e}_k, \\ T_{\mathbf{e}, \phi} &\equiv \sum_{k=0}^{\infty} e_k \otimes \bar{\phi}_k, \end{aligned}$$

where the tensor $x \otimes \bar{y}$ of elements x, y of \mathcal{H} is defined by

$$(x \otimes \bar{y})\xi = (\xi|y)x, \quad \xi \in \mathcal{H}.$$

This is also denoted by the Dirac notation $|x \rangle \langle y|$. Here we use the notation $x \otimes \bar{y}$.

In Section 2, we investigate the relationship between the operator T_e and the operators $T_{\phi,e}$ and $T_{e,\phi}$. The operator $T_{e,\phi}$ is always closed, however $D(T_{\phi,e}^*)$ is not necessarily dense in \mathcal{H} , equivalently, T_e and $T_{\phi,e}$ are not necessarily closable. Indeed, it is shown that the following statements are equivalent:

- (i) T_e is closable.
- (ii) $T_{\phi,e}$ is closable.
- (iii) $D(T_{e,\phi}) = D(\phi) \equiv \{x \in \mathcal{H}; \sum_{k=0}^{\infty} |(x|\phi_k)|^2 < \infty\}$ is dense in \mathcal{H} .

If this holds, then $\bar{T}_e = \bar{T}_{\phi,e} = (T_{e,\phi})^*$.

Furthermore we investigate the relationships between the notion of biorthogonal pairs and the operators $T_{\phi,e}$, $T_{e,\phi}$. Indeed, if $D(\phi)$ is dense in \mathcal{H} , then $\bar{T}_{\phi,e}$ has an inverse and $\bar{T}_{\phi,e}^{-1} \subset T_{e,\psi} = (T_{\psi,e})^*$. However, $D(\bar{T}_{\phi,e}^{-1})$ is not dense in \mathcal{H} in general. And so we may give the conditions under what $D(\bar{T}_{\phi,e}^{-1})$ is dense in \mathcal{H} . In detail, the following statements are equivalent:

- (i) $D_\phi \equiv \text{Span}\{\phi_n\}$ is dense in \mathcal{H} .
- (ii) $T_{\phi,e}$ is closable and $\bar{T}_{\phi,e}$ has a densely defined inverse.
- (iii) $T_{\phi,e}^* (= T_{e,\phi})$ has a densely defined inverse.

If this holds, then $T_{e,\phi}^{-1} = (\bar{T}_{\phi,e})^*$.

In Section 3, we first investigate the relationship between semi-regular biorthogonal pairs and generalized Riesz bases. In Definition 2.1 in Ref [1], the author has defined the notion of generalized Riesz bases under the assumption that D_ϕ and D_ψ are dense in \mathcal{H} , and has shown that if $(\{\phi_n\}, \{\psi_n\})$ is a regular biorthogonal pair, then both $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases. In this section, we redefine the notion of generalized Riesz bases, that is, D_ϕ and D_ψ are not necessarily dense in \mathcal{H} and show that if $(\{\phi_n\}, \{\psi_n\})$ is a semi-regular biorthogonal pair, then both $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases. This result improves the results of Ref. [1, 2, 3]. Furthermore, we have the following results:

- (i) If $(\{\phi_n\}, \{\psi_n\})$ is a regular biorthogonal pair, then for any ONB $e = \{e_n\}$ in \mathcal{H} , $\bar{T}_{\phi,e}$ (resp. $\bar{T}_{\psi,e}$) is the minimum among constructing operators of the generalized Riesz basis $\{\phi_n\}$ (resp. $\{\psi_n\}$) and $T_{e,\psi}^{-1}$ (resp. $T_{e,\phi}^{-1}$) is the

maximum among constructing operator of $\{\phi_n\}$ (resp. $\{\psi_n\}$). Furthermore, any closed operator T (resp. K) satisfying $\bar{T}_{\phi,e} \subset T \subset T_{e,\psi}^{-1}$ (resp. $\bar{T}_{\psi,e} \subset K \subset T_{e,\phi}^{-1}$) is a constructing operator for $\{\phi_n\}$ (resp. $\{\psi_n\}$).

(ii) If $D(\phi)$ and D_ϕ are dense in \mathcal{H} , then $\bar{T}_{\phi,e}$ (resp. $T_{e,\phi}^{-1}$) is the minimum (resp. the maximum) among constructing operators of $\{\phi_n\}$ (resp. $\{\psi_n\}$).

(iii) If $D(\psi)$ and D_ψ are dense in \mathcal{H} , then $\bar{T}_{\psi,e}$ (resp. $T_{e,\psi}^{-1}$) is the minimum (resp. the maximum) among constructing operators of $\{\psi_n\}$ (resp. $\{\phi_n\}$).

We study the physical operators defined by the operators $T_{\phi,e}$, $T_{e,\phi}$, $T_{\psi,e}$ and $T_{e,\psi}$ and an ONB $e = \{e_n\}$. If $D(\phi)$ and D_ϕ are dense in \mathcal{H} , then lowering, raising and number operators $A_{\phi,e}$, $B_{\phi,e}$ and $N_{\phi,e}$ for $\{\phi_n\}$ are defined, respectively, and raising, lowering and number operators $A_{e,\phi}$, $B_{e,\phi}$ and $N_{e,\phi}$ for $\{\psi_n\}$ are defined, respectively. Furthermore, if $D(\psi)$ and D_ψ are dense in \mathcal{H} , then lowering, raising and number operators $A_{\psi,e}$, $B_{\psi,e}$ and $N_{\psi,e}$ for $\{\psi_n\}$ are defined, respectively, and raising, lowering and number operators $A_{e,\psi}$, $B_{e,\psi}$ and $N_{e,\psi}$ for $\{\phi_n\}$ are defined, respectively. These operators connect with *quasi-hermitian quantum mechanics*, and its relatives. [10, 7, 5] Many researchers have investigated such operators mathematically. [1, 3, 2, 4]

In Section 4, we shall show a method of constructing a semi-regular biorthogonal pair based on the following commutation rule under some assumptions. Here, the commutation rule is that a pair of operators a and b acting on a Hilbert space \mathcal{H} satisfying

$$ab - ba = I.$$

The author has given assumptions to construct the regular biorthogonal pair in Ref. [3]. Indeed, the assumptions in Ref. [3] coincide with the definition of pseudo-bosons as originally given in Ref. [8]. We shall give some assumptions to construct the semi-regular biorthogonal pair that connect with the definition of pseudo-bosons, and show that by using the results in Section 3 and Ref. [3], if $D(\phi)$ and D_ϕ are dense in \mathcal{H} , then we may construct new pseudo-bosonic operators $\{A_{\phi,e}, B_{\phi,e}, A_{e,\phi}, B_{e,\phi}\}$ and if $D(\psi)$ and D_ψ are dense in \mathcal{H} , then we may construct a new pseudo-bosonic operators $\{A_{\psi,e}, B_{\psi,e}, A_{e,\psi}, B_{e,\psi}\}$. Furthermore, we investigate the relationship between pseudo-bosonic operators $\{a, b, a^\dagger, b^\dagger\}$ satisfying some assumptions and the operators $\{A_{\phi,e}, B_{\phi,e}, A_{e,\phi}, B_{e,\phi}\}$ and $\{A_{\psi,e}, B_{\psi,e}, A_{e,\psi}, B_{e,\psi}\}$.

This article is organized as follows. In Section 2, we define new operators $T_{\phi,e}$ and $T_{e,\phi}$ and study the property of these operators. Furthermore, we study the relationship between the operator T_e and the operators $T_{\phi,e}$ and $T_{e,\phi}$. In Section 3, we investigate the relationship between semi-regular biorthogonal pairs and generalized Riesz bases and give the physical operators defined by the operators $T_{\phi,e}$, $T_{e,\phi}$, $T_{\psi,e}$ and $T_{e,\psi}$ and an ONB $e = \{e_n\}$. In Section 4, we introduce a method of constructing a semi-regular biorthogonal pair based on the pseudo-bosonic operators $\{a, b, a^\dagger, b^\dagger\}$ under some assumptions and we investigate the relationship between pseudo-bosonic operators satisfying some assumptions and the physical operators $\{A_{\phi,e}, B_{\phi,e}, A_{e,\phi}, B_{e,\phi}\}$ and $\{A_{\psi,e}, B_{\psi,e}, A_{e,\psi}, B_{e,\psi}\}$. In Section 5, we describe future issue with respect to biorthogonal pairs $(\{\phi_n\}, \{\psi_n\})$ and generalized Riesz bases.

2 Some operators defined by biorthogonal sequences and ONB

Let \mathcal{H} be a Hilbert space with inner product $(\cdot|\cdot)$. We consider the following operators in \mathcal{H} defined by a sequence $\{\phi_n\}$ in a Hilbert space \mathcal{H} and an ONB $e = \{e_n\}$ in \mathcal{H} :

$$\begin{aligned} T_{\phi,e} &\equiv \sum_{k=0}^{\infty} \phi_k \otimes \bar{e}_k, \\ T_{e,\phi} &\equiv \sum_{k=0}^{\infty} e_k \otimes \bar{\phi}_k. \end{aligned}$$

In Ref. [2], the author have defined an operator T_e on $D_e \equiv \text{Span}\{e_n\}$ by

$$T_e \left(\sum_{k=0}^n \alpha_k e_k \right) = \sum_{k=0}^n \alpha_k \phi_k.$$

For the operators $T_{\phi,e}$, $T_{e,\phi}$ and T_e we have the following

Lemma 2.1. The following statements hold.

(1) $T_{\phi,e}$ is a densely defined linear operator in \mathcal{H} such that

$$T_{\phi,e} \supset T_e \quad \text{and} \quad T_{\phi,e} e_n = \phi_n, \quad n = 0, 1, \dots$$

(2)

$$D(T_{\mathbf{e},\phi}) = D(\phi) \equiv \left\{ x \in \mathcal{H}; \sum_{k=0}^{\infty} |(x|\phi_k)|^2 < \infty \right\} \quad \text{and} \quad T_{\mathbf{e}}^* = T_{\phi,\mathbf{e}}^* = T_{\mathbf{e},\phi}.$$

Proof. The statements (1) and (2) are easily proved by the definitions of $T_{\phi,\mathbf{e}}$, $T_{\mathbf{e},\phi}$ and $T_{\mathbf{e}}$.

By Lemma 2.1, (2), $T_{\mathbf{e},\phi}$ is closed. However $D(T_{\phi,\mathbf{e}}^*)$ is not necessarily dense in \mathcal{H} , equivalently, $T_{\mathbf{e}}$ and $T_{\phi,\mathbf{e}}$ are not necessarily closable. Thus we investigate the conditions under what $T_{\phi,\mathbf{e}}$ is closable.

Lemma 2.2. *The following statements are equivalent:*

- (i) $T_{\mathbf{e}}$ is closable.
- (ii) $T_{\phi,\mathbf{e}}$ is closable.
- (iii) $D(\phi)$ is dense in \mathcal{H} .

If this holds, then

$$\bar{T}_{\mathbf{e}} = \bar{T}_{\phi,\mathbf{e}} = (T_{\mathbf{e},\phi})^*.$$

Proof. This follows from Lemma 2.1, (2).

Next we study the relationships between the notion of biorthogonal pairs and the operators $T_{\phi,\mathbf{e}}$, $T_{\mathbf{e},\phi}$. Then we have the following statements.

Lemma 2.3. *Suppose that $(\{\phi_n\}, \{\psi_n\})$ is a biorthogonal pair such that $D(\phi)$ is dense in \mathcal{H} , then $\bar{T}_{\phi,\mathbf{e}}$ has an inverse and $\bar{T}_{\phi,\mathbf{e}}^{-1} \subset T_{\mathbf{e},\psi} = (T_{\psi,\mathbf{e}})^*$.*

Proof. By the definitions of $T_{\phi,\mathbf{e}}$ and $T_{\mathbf{e},\psi}$, we have

$$T_{\mathbf{e},\psi} T_{\phi,\mathbf{e}} e_n = T_{\mathbf{e},\psi} \phi_n = e_n, \quad n = 0, 1, \dots$$

Hence we have

$$T_{\mathbf{e},\psi} T_{\phi,\mathbf{e}} = I \quad \text{on} \quad D_{\mathbf{e}}.$$

Thus we have

$$T_{\mathbf{e},\psi} \bar{T}_{\phi,\mathbf{e}} = I.$$

This completes the proof.

In general, $D(\bar{T}_{\phi,e}^{-1})$ is not necessarily dense in \mathcal{H} . We investigate the conditions under what $D(\bar{T}_{\phi,e}^{-1})$ is dense in \mathcal{H} .

Lemma 2.4. Suppose that $(\{\phi_n\}, \{\psi_n\})$ is a biorthogonal pair such that $D(\phi)$ is dense in \mathcal{H} . Then the following statements are equivalent:

- (i) $D_\phi \equiv \text{Span}\{\phi_n\}$ is dense in \mathcal{H} .
- (ii) $T_{\phi,e}$ is closable and $\bar{T}_{\phi,e}$ has a densely defined inverse.
- (iii) $T_{\phi,e}^* (= T_{e,\phi})$ has a densely defined inverse.

If this holds, then $T_{e,\phi}^{-1} = (\bar{T}_{\phi,e}^{-1})^$.*

Proof. (i) \Rightarrow (ii) Since $D(\phi)$ is dense in \mathcal{H} , by Lemma 2.2 and Lemma 2.3 we have $T_{\phi,e}$ is closable and $\bar{T}_{\phi,e}$ has an inverse. Furthermore, since $D(\bar{T}_{\phi,e}^{-1}) = \bar{T}_{\phi,e} D(\bar{T}_{\phi,e}) \supset D_\phi$ and D_ϕ is dense in \mathcal{H} , $\bar{T}_{\phi,e}^{-1}$ is densely defined. (ii) \Rightarrow (iii) By Ref. [2], Lemma 2.2 and Lemma 2.3, we have

$$(\bar{T}_{\phi,e}^{-1})^* = (\bar{T}_{\phi,e}^*)^{-1} = (T_{e,\phi})^{-1}.$$

Hence we have

$$D((\bar{T}_{\phi,e}^*)^{-1}) = T_{e,\phi} D(T_{e,\phi}) \supset T_{e,\phi} D_\psi = D_e.$$

Thus we have $(T_{\phi,e}^*)^{-1}$ is densely defined.

(iii) \Rightarrow (i) Take an arbitrary $x \in D_\phi^\perp$. Then,

$$0 = (\phi_n | x) = (T_{\phi,e} e_n | x) = (T_e e_n | x), \quad n = 0, 1, \dots$$

Hence, by Lemma 2.1, (2) we have

$$x \in D(T_e^*) = D(T_{\phi,e}^*) \quad \text{and} \quad T_e^* x = T_{\phi,e}^* x = 0.$$

By (iii), it follows that

$$x = (T_{\phi,e}^*)^{-1} T_{\phi,e}^* x = 0.$$

Thus, D_ϕ is dense in \mathcal{H} . This completes the proof.

Similarly we have the following statements.

Lemma 2.5. Suppose $(\{\phi_n\}, \{\psi_n\})$ is a biorthogonal pair such that $D(\psi)$ is dense in \mathcal{H} . Then the following statements are equivalent:

- (i) $D_\psi \equiv \text{Span}\{\psi_n\}$ is dense in \mathcal{H} .*
- (ii) $T_{\psi, \mathbf{e}}$ is closable and $\bar{T}_{\psi, \mathbf{e}}$ has a densely defined inverse.*
- (iii) $T_{\psi, \mathbf{e}}^* (= T_{\mathbf{e}, \psi})$ has a densely defined inverse.*

If this holds, then $T_{\mathbf{e}, \psi}^{-1} = (\bar{T}_{\psi, \mathbf{e}}^{-1})^$.*

Proof. This is shown similarly to Lemma 2.4.

3 Semi-regular biorthogonal pairs and generalized Riesz bases

In Ref. [1], the author has defined the notion of generalized Riesz bases. First we redefine the notion of generalized Riesz bases.

Definition 3.1. If there exists a densely defined closed operator T in \mathcal{H} with a densely defined inverse and there exists an ONB $\mathbf{e} = \{e_n\}$ in \mathcal{H} such that

$$\{e_n\} \subset D(T) \cap D((T^{-1})^*) \quad \text{and} \quad Te_n = \phi_n, \quad n = 0, 1, \dots,$$

then a sequence $\{\phi_n\}$ in \mathcal{H} is called a generalized Riesz basis with a constructing pair (\mathbf{e}, T) .

Here, we delete the conditions of Definition 2.1, (ii) and (iii) in Ref. [1], that is, D_ϕ and D_ψ are not necessarily dense in \mathcal{H} . Then we have the following

Lemma 3.2. Let $\{\phi_n\}$ be a generalized Riesz basis. Then, we have the following statements.

- (1) T^* has a densely defined inverse and $(T^*)^{-1} = (T^{-1})^*$.*

(2) $\psi_n \equiv (T^{-1})^* e_n$, $n = 0, 1, \dots$. Then, $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal and $(T^{-1})^*$ is a densely defined closed operator in \mathcal{H} with densely defined inverse T^* . Hence $\{\psi_n\}$ is a generalized Riesz basis with a constructing pair $(\mathbf{e}, (T^{-1})^*)$.

(3) $D(\phi) \cap D(\psi)$ is dense in \mathcal{H} .

Proof. (1) and (2) are easily shown.

(3) We first show that

$$D(T^*) \subset D(\phi) \quad \text{and} \quad R(T) = D(T^{-1}) \subset D(\psi). \quad (2.1)$$

Indeed, this follows from

$$\begin{aligned} \sum_{k=0}^{\infty} |(x|\phi_k)|^2 &= \sum_{k=0}^{\infty} |(T^*x|e_k)|^2 \\ &= \|T^*x\|^2, \quad x \in D(T^*) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} |(y|\psi_k)|^2 &= \sum_{k=0}^{\infty} |(T^{-1}y|e_k)|^2 \\ &= \|T^{-1}y\|^2, \quad y \in D(T^{-1}). \end{aligned}$$

Since $D(T^*)$ and $R(T)$ are dense in \mathcal{H} , it follows that $D(\phi)$ and $D(\psi)$ are dense in \mathcal{H} . Next we show that $D(\phi) \cap D(\psi)$ is dense in \mathcal{H} . Take an arbitrary $x \in D(T)$. Let $|T| = \int_0^\infty \lambda dE_T(\lambda)$ be the spectral resolution of the absolute $|T| \equiv (T^*T)^{\frac{1}{2}}$ of T . Then we have $TE_T(n)x \in D(T^*) \cap R(T)$, $n = 0, 1, \dots$ and $\lim_{n \rightarrow \infty} TE_T(n)x = Tx$. Hence $D(T^*) \cap R(T)$ is dense in $R(T)$, and since $R(T)$ is dense in \mathcal{H} , it follows from (2.1) that $D(\phi) \cap D(\psi)$ is dense in \mathcal{H} . This completes the proof.

In Ref. [2], we have shown that if $(\{\phi_n\}, \{\psi_n\})$ is a regular biorthogonal pair, then both $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases. In order to generalize this result, we define the notion of semi-regular biorthogonal pair as follows:

Definition 3.3. A pair $(\{\phi_n\}, \{\psi_n\})$ of biorthogonal sequences in \mathcal{H} is said to be semi-regular if either $D(\phi)$ and D_ϕ are dense in \mathcal{H} or $D(\psi)$ and D_ψ

are dense in \mathcal{H} .

We give a concrete example[6] of semi-regular and non regular biorthogonal bases. Let $\{e_n\}$ be an ONB in \mathcal{H} and put $\phi_n = e_n + e_0$ and $\psi_n = e_n$, $n = 1, 2, \dots$. Then it is easily shown that $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal bases such that D_ϕ and $D(\phi)$ are dense in \mathcal{H} , but D_ψ is not dense in \mathcal{H} . We show that if $(\{\phi_n\}, \{\psi_n\})$ is a semi-regular biorthogonal pair, then both $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases. In detail, we have the following

Theorem 3.4. Let $\{\phi_n\}$ and $\{\psi_n\}$ be biorthogonal sequences in \mathcal{H} , and let $\mathbf{e} = \{e_n\}$ be an arbitrary ONB in \mathcal{H} . Then the following statements hold:

(1) *Suppose that $(\{\phi_n\}, \{\psi_n\})$ is a regular biorthogonal pair. Then $\{\phi_n\}$ (resp. $\{\psi_n\}$) is a generalized Riesz basis with constructing pairs $(\mathbf{e}, \bar{T}_{\phi, \mathbf{e}})$ and $(\mathbf{e}, T_{\mathbf{e}, \psi}^{-1})$ (resp. $(\mathbf{e}, \bar{T}_{\psi, \mathbf{e}})$ and $(\mathbf{e}, T_{\mathbf{e}, \phi}^{-1})$), and $\bar{T}_{\phi, \mathbf{e}}$ (resp. $\bar{T}_{\psi, \mathbf{e}}$) is the minimum among constructing operators of the generalized Riesz basis $\{\phi_n\}$ (resp. $\{\psi_n\}$), and $T_{\mathbf{e}, \psi}^{-1}$ (resp. $T_{\mathbf{e}, \phi}^{-1}$) is the maximal among constructing operators of $\{\phi_n\}$ (resp. $\{\psi_n\}$). Furthermore, any closed operator T (resp. K) satisfying $\bar{T}_{\phi, \mathbf{e}} \subset T \subset T_{\mathbf{e}, \psi}^{-1}$ (resp. $\bar{T}_{\psi, \mathbf{e}} \subset K \subset T_{\mathbf{e}, \phi}^{-1}$) is a constructing operator for $\{\phi_n\}$ (resp. $\{\psi_n\}$).*

(2) *Suppose that $D(\phi)$ and D_ϕ are dense in \mathcal{H} . Then $\{\phi_n\}$ (resp. $\{\psi_n\}$) is a generalized Riesz basis with a constructing pair $(\mathbf{e}, \bar{T}_{\phi, \mathbf{e}})$ (resp. $(\mathbf{e}, T_{\mathbf{e}, \phi}^{-1})$) and the constructing operator $\bar{T}_{\phi, \mathbf{e}}$ (resp. $T_{\mathbf{e}, \phi}^{-1}$) is the minimum (resp. the maximum) among constructing operators of $\{\phi_n\}$ (resp. $\{\psi_n\}$).*

(3) *Suppose that $D(\psi)$ and D_ψ are dense in \mathcal{H} . Then $\{\psi_n\}$ (resp. $\{\phi_n\}$) is a generalized Riesz basis with a constructing pair $(\mathbf{e}, \bar{T}_{\psi, \mathbf{e}})$ (resp. $(\mathbf{e}, T_{\mathbf{e}, \psi}^{-1})$) and the constructing operator $\bar{T}_{\psi, \mathbf{e}}$ (resp. $T_{\mathbf{e}, \psi}^{-1}$) is the minimum (resp. the maximum) among constructing operators of $\{\psi_n\}$ (resp. $\{\phi_n\}$).*

Proof. Let $\mathbf{e} = \{e_n\}$ be any ONB in \mathcal{H} .

(1) Since $D(\phi)$ is dense in \mathcal{H} , it follows from Lemma 2.3 that $\bar{T}_{\phi, \mathbf{e}}$ has an inverse. Since D_ϕ is also dense in \mathcal{H} , it follows from Lemma 2.4 that the inverse $\bar{T}_{\phi, \mathbf{e}}^{-1}$ of $\bar{T}_{\phi, \mathbf{e}}$ is densely defined. Furthermore, since $\bar{T}_{\phi, \mathbf{e}}^* \psi_n = T_{\mathbf{e}, \phi} \psi_n = e_n$, $n = 0, 1, \dots$, we have $\mathbf{e} \subset D((\bar{T}_{\phi, \mathbf{e}}^*)^{-1}) = D(T_{\mathbf{e}, \phi}^{-1})$. Thus $\{\phi_n\}$ is a generalized Riesz basis with a constructing pair $(\mathbf{e}, \bar{T}_{\phi, \mathbf{e}})$, and $\{\psi_n\}$ is a generalized Riesz basis with a constructing pair $(\mathbf{e}, T_{\mathbf{e}, \phi}^{-1})$. Similarly, $\{\psi_n\}$ is a general-

ized Riesz basis with a constructing pair $(\mathbf{e}, \bar{T}_{\psi, \mathbf{e}})$, and $\{\phi_n\}$ is a generalized Riesz basis with a constructing pair $(\mathbf{e}, T_{\mathbf{e}, \psi}^{-1})$. Hence $\{\phi_n\}$ (resp. $\{\psi_n\}$) is a generalized Riesz basis with constructing pairs $(\mathbf{e}, \bar{T}_{\phi, \mathbf{e}})$ and $(\mathbf{e}, T_{\mathbf{e}, \psi}^{-1})$ (resp. $(\mathbf{e}, \bar{T}_{\psi, \mathbf{e}})$ and $(\mathbf{e}, T_{\mathbf{e}, \phi}^{-1})$).

Take an arbitrary constructing operator T of the generalized Riesz basis $\{\phi_n\}$. Since $Te_n = \phi_n$ and $(T^{-1})^*e_n = \psi_n$, $n = 0, 1, \dots$, we have $\bar{T}_{\phi, \mathbf{e}} \subset T$ and $\bar{T}_{\psi, \mathbf{e}} \subset (T^{-1})^*$, which implies that $T^{-1} \subset T_{\psi, \mathbf{e}}^* = T_{\mathbf{e}, \psi}$. Hence, we have $T \subset T_{\mathbf{e}, \psi}^{-1}$. Thus, $\bar{T}_{\phi, \mathbf{e}}$ and $T_{\mathbf{e}, \psi}^{-1}$ are the minimum and the maximum among constructing operators of $\{\phi_n\}$, respectively. Furthermore, suppose that T is a closed operator in \mathcal{H} such that $\bar{T}_{\phi, \mathbf{e}} \subset T \subset T_{\mathbf{e}, \psi}^{-1}$. Then, since $D(T) \supset D_{\mathbf{e}}$, $TD(T) \supset T_{\phi, \mathbf{e}}D_{\mathbf{e}} = \{\phi_n\}$ and $D((T^*)^{-1}) \supset D(T_{\psi, \mathbf{e}}) \supset D_{\mathbf{e}}$, it follows that T is a constructing operator for $\{\phi_n\}$. Similar results for $\{\psi_n\}$ are obtained. The statements (2) and (3) are shown similarly to (1). This completes the proof.

Remark. Theorem 3.4 means the following:

(1) Suppose $D(\phi)$ and D_{ϕ} (resp. $D(\psi)$ and D_{ψ}) are dense in \mathcal{H} . Even if D_{ψ} (resp. D_{ϕ}) is not dense in \mathcal{H} , $\{\psi_n\}$ (resp. $\{\phi_n\}$) becomes a generalized Riesz basis.

(2) Suppose that $D(\phi)$ and D_{ϕ} are dense in \mathcal{H} , but D_{ψ} is not dense in \mathcal{H} . As shown in Theorem 3.4, $\bar{T}_{\phi, \mathbf{e}}$ is the minimum among constructing operators of $\{\phi_n\}$, however the maximal constructing operator of $\{\phi_n\}$ does not necessarily exist because $T_{\mathbf{e}, \psi}^{-1}$ is not a constructing operator of $\{\phi_n\}$ different to the case of regular biorthogonal pair. Furthermore, $T_{\mathbf{e}, \phi}^{-1}$ is the maximum among constructing operators of $\{\psi_n\}$, however the minimal constructing operator of $\{\psi_n\}$ does not necessarily exist because $\bar{T}_{\psi, \mathbf{e}}$ is not a constructing operator of $\{\psi_n\}$. Similar results for the case that $D(\psi)$ and D_{ψ} are dense in \mathcal{H} , but D_{ϕ} is not dense in \mathcal{H} are obtained.

By Theorem 3.4, Ref. [1] and [3], we can define the physical operators as follows:

(1) Suppose $D(\phi)$ and D_ϕ are dense in \mathcal{H} . Then, we put

$$\begin{aligned}
A_{\phi,e} &= \bar{T}_{\phi,e} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right) \bar{T}_{\phi,e}^{-1}, \\
B_{\phi,e} &= \bar{T}_{\phi,e} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right) \bar{T}_{\phi,e}^{-1}, \\
N_{\phi,e} &= \bar{T}_{\phi,e} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_{k+1} \right) \bar{T}_{\phi,e}^{-1}, \\
\\
A_{e,\phi} &= T_{e,\phi}^{-1} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right) T_{e,\phi}, \\
B_{e,\phi} &= T_{e,\phi}^{-1} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right) T_{e,\phi}, \\
N_{e,\phi} &= T_{e,\phi}^{-1} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_{k+1} \right) T_{e,\phi}.
\end{aligned}$$

(2) Suppose $D(\psi)$ and D_ψ are dense in \mathcal{H} . Then, we put

$$\begin{aligned}
A_{\psi,e} &= \bar{T}_{\psi,e} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right) \bar{T}_{\psi,e}^{-1}, \\
B_{\psi,e} &= \bar{T}_{\psi,e} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right) \bar{T}_{\psi,e}^{-1}, \\
N_{\psi,e} &= \bar{T}_{\psi,e} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_{k+1} \right) \bar{T}_{\psi,e}^{-1}, \\
\\
A_{e,\psi} &= T_{e,\psi}^{-1} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right) T_{e,\psi}, \\
B_{e,\psi} &= T_{e,\psi}^{-1} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right) T_{e,\psi}, \\
N_{e,\psi} &= T_{e,\psi}^{-1} \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_{k+1} \right) T_{e,\psi}.
\end{aligned}$$

Then we have the following

Theorem 3.5. The following statements hold.

(1) *Suppose that $D(\phi)$ and D_ϕ are dense in \mathcal{H} . Then we have*

$$A_{\phi,e}\phi_n = \begin{cases} 0 & , n = 0, \\ \sqrt{n}\phi_{n-1}, & , n = 1, 2, \dots, \end{cases}$$

$$B_{\phi,e}\phi_n = \sqrt{n+1}\phi_{n+1} \quad , n = 0, 1, \dots,$$

$$N_{\phi,e}\phi_n = n\phi_n,$$

$$A_{e,\phi}\psi_n = \sqrt{n+1}\psi_{n+1} \quad , n = 0, 1, \dots,$$

$$B_{e,\phi}\psi_n = \begin{cases} 0 & , n = 0, \\ \sqrt{n}\psi_{n-1}, & , n = 1, 2, \dots, \end{cases}$$

$$N_{e,\phi}\psi_n = n\psi_n.$$

Hence $A_{\phi,e}$, $B_{\phi,e}$ and $N_{\phi,e}$ are lowering, raising and number operators for $\{\phi_n\}$, respectively, and $A_{e,\phi}$, $B_{e,\phi}$ and $N_{e,\phi}$ are raising, lowering and number operators for $\{\psi_n\}$, respectively.

(2) Suppose that $D(\psi)$ and D_ψ are dense in \mathcal{H} . Then we have

$$A_{\psi,e}\psi_n = \begin{cases} 0 & , n = 0, \\ \sqrt{n}\psi_{n-1}, & , n = 1, 2, \dots, \end{cases}$$

$$B_{\psi,e}\psi_n = \sqrt{n+1}\psi_{n+1} \quad , n = 0, 1, \dots,$$

$$N_{\psi,e}\psi_n = n\psi_n,$$

$$A_{e,\psi}\phi_n = \sqrt{n+1}\phi_{n+1} \quad , n = 0, 1, \dots,$$

$$B_{e,\psi}\phi_n = \begin{cases} 0 & , n = 0, \\ \sqrt{n}\phi_{n-1}, & , n = 1, 2, \dots, \end{cases}$$

$$N_{e,\psi}\phi_n = n\phi_n.$$

Hence $A_{\psi,e}$, $B_{\psi,e}$ and $N_{\psi,e}$ are lowering, raising and number operators for $\{\psi_n\}$, respectively, and $A_{e,\psi}$, $B_{e,\psi}$ and $N_{e,\psi}$ are raising, lowering and number operators for $\{\phi_n\}$, respectively.

Remark.

(i) In case of (1), since

$$\begin{aligned} A_{e,\phi} &= (T_{\phi,e}^{-1})^* \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right)^* T_{\phi,e}^*, \\ B_{e,\phi} &= (T_{\phi,e}^{-1})^* \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right)^* T_{\phi,e}^*, \\ N_{e,\phi} &= (T_{\phi,e}^{-1})^* \left(\sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_{k+1} \right)^* T_{\phi,e}^*, \end{aligned}$$

the author has denoted A_e^\dagger , B_e^\dagger and N_e^\dagger in Ref. [3].

(ii) Suppose that $D(\phi)$ and D_ϕ are dense in \mathcal{H} . Then the number operators $N_{\phi,e}$ and $N_{e,\phi} (\equiv N_{\phi,e}^\dagger)$ for $\{\phi_n\}$ and $\{\psi_n\}$, respectively have the relation: $(\bar{T}_{\phi,e}^{-1})^* \bar{T}_{\phi,e}^{-1} N_{\phi,e} = N_{\phi,e}^\dagger (\bar{T}_{\phi,e}^{-1})^* \bar{T}_{\phi,e}^{-1}$. This is called that $N_{\phi,e}$ is a quasi-Hermitian operator [11, 12, 9] and positive self-adjoint operator $(\bar{T}_{\phi,e}^{-1})^* \bar{T}_{\phi,e}^{-1}$

is often called a metric operator for the quasi-Hermitian operator $N_{\phi,e}$. Suppose that $D(\psi)$ and D_ψ are dense in \mathcal{H} . Then the number operator $N_{e,\psi}$ is a quasi-Hermitian operator for the metric operator $(\bar{T}_{\psi,e}^{-1})^* \bar{T}_{\psi,e}^{-1}$. The results on generalized Riesz bases is related to the problem of finding metric operators for quasi-Hermitian operators.

4 Semi-regular biorthogonal pairs and Psuedo-bosonic operators

In this section, we introduce a method of constructing a semi-regular biorthogonal pair based on the following commutation rule under some assumptions. Here, the commutation rule is that a pair of operators a and b acting on a Hilbert space \mathcal{H} with inner product $(\cdot|\cdot)$ satisfies

$$ab - ba = I.$$

In particular, this collapses to the canonical commutation rule (CCR) if $b = a^\dagger$. In Ref. [3] the author has shown assumptions to construct the regular biorthogonal pair. Indeed, the assumptions in Ref. [3] coincide with the definition of pseudo-bosons as originally given in Ref. [8], where in the recent literature many researchers have investigated. [6, 8, 7, 10, 13]. In this section, we introduce that some assumptions to construct the semi-regular biorthogonal pair connect with the definition of pseudo-bosons. At first, we construct semi-regular biorthogonal pairs on the above commutation rule. We assume the following statements:

Assumption 1. There exists a non-zero element ϕ_0 of \mathcal{H} such that

- (i) $a\phi_0 = 0$,
- (ii) $\phi_0 \in D^\infty(b) \equiv \cap_{k=0}^\infty D(b^k)$,
- (iii) $b^n \phi_0 \in D(a)$, $n = 0, 1, \dots$.

Then, we may define a sequence $\{\phi_n\}$ in \mathcal{H} by

$$\begin{aligned} \phi_n &\equiv \frac{1}{\sqrt{n!}} b^n \phi_0, \quad n = 0, 1, \dots \\ &= \frac{1}{\sqrt{n}} b \phi_{n-1}, \quad n = 1, 2, \dots \end{aligned}$$

Furthermore, we have the following

Proposition 4.1. The following statements hold.

(1) $b^n \phi_0 \in D(a^m)$ and

$$a^m b^n \phi_0 = \begin{cases} {}_n P_m b^{n-m} \phi_0 & , m \leq n, \\ 0 & , m > n. \end{cases}$$

(2) $\phi_n \in D(N^m)$ and $N^m \phi_n = n^m \phi_n$, $n, m = 0, 1, \dots$. In particular, $N \phi_n = n \phi_n$, $n = 0, 1, \dots$.

(3)

$$\begin{aligned} a \phi_n &= \begin{cases} 0 & , n = 0, \\ \sqrt{n} \phi_{n-1}, & , n = 1, 2, \dots, \end{cases} \\ b \phi_n &= \sqrt{n+1} \phi_{n+1} \quad , n = 0, 1, \dots. \end{aligned}$$

Proof. These proofs follow from Ref. [3].

Assumption 2. There exists a non-zero element ψ_0 of \mathcal{H} such that

- (i) $b^\dagger \psi_0 = 0$,
- (ii) $\psi_0 \in D^\infty(a^\dagger) \equiv \cap_{k=0}^\infty D((a^\dagger)^k)$,
- (iii) $(a^\dagger)^n \psi_0 \in D(b^\dagger)$, $n = 0, 1, \dots$.

Then, we may define a sequence $\{\psi_n\}$ in \mathcal{H} by

$$\begin{aligned} \psi_n &\equiv \frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_0, \quad n = 0, 1, \dots \\ &= \frac{1}{\sqrt{n}} a^\dagger \psi_{n-1}, \quad n = 1, 2, \dots. \end{aligned}$$

And we put an operator $N^\dagger \equiv a^\dagger b^\dagger$. Furthermore we have the following

Proposition 4.2. The following statements hold.

(1) $(a^\dagger)^n \psi_0 \in D((b^\dagger)^m)$ and

$$(b^\dagger)^m (a^\dagger)^n \psi_0 = \begin{cases} {}_n P_m (a^\dagger)^{n-m} \psi_0 & , m \leq n, \\ 0 & , m > n. \end{cases}$$

(2) $\psi_n \in D((N^\dagger)^m)$ and $(N^\dagger)^m \psi_n = n^m \psi_n$, $n, m = 0, 1, \dots$. In particular, $N^\dagger \psi_n = n \psi_n$, $n = 0, 1, \dots$.
(3)

$$a^\dagger \psi_n = \sqrt{n+1} \psi_{n+1}, \quad n = 0, 1, \dots,$$

$$b^\dagger \psi_n = \begin{cases} 0 & , n = 0, \\ \sqrt{n} \psi_{n-1}, & , n = 1, 2, \dots \end{cases}$$

Proof. These proofs follow from Ref. [3].

The above Assumption 1 and Assumption 2 coincide with the assumptions of Ref. [3]. We weaken the assumption of Ref. [3] to the next assumption in order to construct semi-regular biorthogonal pairs.

Assumption 3.

Either $D(\phi)$ and D_ϕ are dense in \mathcal{H} or $D(\psi)$ and D_ψ are dense in \mathcal{H} .

Then, if a pair of operators a and b acting on \mathcal{H} satisfies Assumption 1-3, $(\{\phi_n\}, \{\psi_n\})$ becomes a semi-regular biorthogonal pair.

By Section 2, Section 3 and Ref. [2], in case of $D(\phi)$ and D_ϕ are dense in \mathcal{H} (resp. $D(\psi)$ and D_ψ are dense in \mathcal{H}), $A_{\phi,e}$, $B_{\phi,e}$ and $N_{\phi,e}$ are lowering, raising and number operators for $\{\phi_n\}$, respectively, and $A_{e,\phi}$, $B_{e,\phi}$ and $N_{e,\phi}$ are raising, lowering and number operators for $\{\psi_n\}$, respectively. (resp. $A_{\psi,e}$, $B_{\psi,e}$ and $N_{\psi,e}$ are lowering, raising and number operators for $\{\psi_n\}$, respectively, and $A_{e,\psi}$, $B_{e,\psi}$ and $N_{e,\psi}$ are raising, lowering and number operators for $\{\phi_n\}$, respectively). And we have

$$\begin{aligned} A_{\phi,e} B_{\phi,e} - B_{\phi,e} A_{\phi,e} &\subset I \quad \text{and} \quad B_{e,\phi} A_{e,\phi} - A_{e,\phi} B_{e,\phi} \subset I. \\ (\text{resp. } A_{\psi,e} B_{\psi,e} - B_{\psi,e} A_{\psi,e} &\subset I \quad \text{and} \quad B_{e,\psi} A_{e,\psi} - A_{e,\psi} B_{e,\psi} \subset I.) \end{aligned}$$

Furthermore, we have the following statements with respect to the operators $A_{\phi,e}$, $B_{\phi,e}$, $A_{e,\phi}$ and $B_{e,\phi}$ (resp. $A_{\psi,e}$, $B_{\psi,e}$, $A_{e,\psi}$ and $B_{e,\psi}$). The proofs are easily shown.

Proposition 4.3. *If $D(\phi)$ and D_ϕ are dense in \mathcal{H} , then the following statements hold.*

(1)

$$\begin{aligned}\phi_n &= \frac{1}{\sqrt{n!}} B_{\phi, e}^n \phi_0, \quad n = 0, 1, \dots, \\ \psi_n &= \frac{1}{\sqrt{n!}} A_{e, \phi}^n \psi_0, \quad n = 0, 1, \dots.\end{aligned}$$

(2)

$$\begin{aligned}A_{\phi, e} D_\phi &= D_\phi, & B_{\phi, e} D_\phi &= D_\phi, \\ &\text{and} \\ A_{e, \phi} D_\psi &= D_\psi, & B_{e, \phi} D_\psi &= D_\psi.\end{aligned}$$

Proposition 4.4. If $D(\psi)$ and D_ψ are dense in \mathcal{H} , then the following statements hold.

(1)

$$\begin{aligned}\psi_n &= \frac{1}{\sqrt{n!}} B_{\psi, e}^n \psi_0, \quad n = 0, 1, \dots, \\ \phi_n &= \frac{1}{\sqrt{n!}} A_{e, \psi}^n \phi_0, \quad n = 0, 1, \dots.\end{aligned}$$

(2)

$$\begin{aligned}A_{\psi, e} D_\psi &= D_\psi, & B_{\psi, e} D_\psi &= D_\psi, \\ &\text{and} \\ A_{e, \psi} D_\phi &= D_\phi, & B_{e, \psi} D_\phi &= D_\phi.\end{aligned}$$

Next we investigate the relationship between pseudo-bosonic operators $\{a, b, a^\dagger, b^\dagger\}$ satisfying Assumption 1-3 and the operators $A_{\phi, e}$, $B_{\phi, e}$, $A_{e, \phi}$ and $B_{e, \phi}$ ($A_{\psi, e}$, $B_{\psi, e}$, $A_{e, \psi}$ and $B_{e, \psi}$).

By Proposition 4.1, Proposition 4.2 and Theorem 3.5 we have the following

Lemma 4.5. The following statements hold.

(1) If $D(\phi)$ and D_ϕ are dense in \mathcal{H} , then $D(a) \cap D(b) \supset D_\phi$,

$$a \lceil_{D_\phi} \subset A_{\phi, e} \quad \text{and} \quad b \lceil_{D_\phi} \subset B_{\phi, e}.$$

(2) If $D(\psi)$ and D_ψ are dense in \mathcal{H} , then $D(a^\dagger) \cap D(b^\dagger) \supset D_\psi$,

$$a^\dagger \upharpoonright_{D_\psi} \subset B_{\psi,e} \quad \text{and} \quad b^\dagger \upharpoonright_{D_\psi} \subset A_{\psi,e}.$$

Proposition 4.6. The following statements hold.

(1) Suppose that $D(\phi)$ is dense in \mathcal{H} and D_ϕ is a core for \bar{a} and \bar{b} , then $\bar{a} \subset \bar{A}_{\phi,e}$ and $\bar{b} \subset \bar{B}_{\phi,e}$. In particular, if $\bar{T}_{\phi,e}^{-1}$ is bounded, then $\bar{a} \subset A_{\phi,e} = \bar{A}_{\phi,e}$ and $\bar{b} \subset B_{\phi,e} = \bar{B}_{\phi,e}$, and if $\bar{T}_{\phi,e}$ is bounded, then $\bar{a} = \bar{A}_{\phi,e}$ and $\bar{b} = \bar{B}_{\phi,e}$.

(2) Suppose that $D(\psi)$ is dense in \mathcal{H} and D_ψ is a core for \bar{a}^\dagger and \bar{b}^\dagger , then $\bar{a}^\dagger \subset \bar{B}_{\psi,e}$ and $\bar{b}^\dagger \subset \bar{A}_{\psi,e}$. In particular, if $\bar{T}_{\psi,e}^{-1}$ is bounded, then $\bar{a}^\dagger \subset B_{\psi,e} = \bar{B}_{\psi,e}$ and $\bar{b}^\dagger \subset A_{\psi,e} = \bar{A}_{\psi,e}$, and if $\bar{T}_{\psi,e}$ is bounded, then $\bar{a}^\dagger = \bar{B}_{\psi,e}$ and $\bar{b}^\dagger = \bar{A}_{\psi,e}$.

Proof. This is shown similarly to Proposition 2.5 in Ref. [3] by using Lemma 4.5.

5 Discussions

As shown in Theorem 3.4, if $(\{\phi_n\}, \{\psi_n\})$ is a semi-regular biorthogonal pair, then $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases, and so the physical operators (lowering, raising and number operators) are constructed. In case that $(\{\phi_n\}, \{\psi_n\})$ is not a semi-regular biorthogonal pair, that is, both D_ϕ and D_ψ are not dense in \mathcal{H} , it is meaningful to consider the following question:

Question. Under what conditions is a biorthogonal pair $(\{\phi_n\}, \{\psi_n\})$ a generalized Riesz basis?

We have estimated that if a biorthogonal pair $(\{\phi_n\}, \{\psi_n\})$ is a \mathcal{D} -quasi basis [6, 5], then $\{\phi_n\}$ and $\{\psi_n\}$ are generalized Riesz bases, where \mathcal{D} is a dense subspace in \mathcal{H} and $(\{\phi_n\}, \{\psi_n\})$ is a \mathcal{D} -quasi basis if

$$(f, g) = \sum_{k=0}^{\infty} (f, \psi_k)(\phi_k, g) = \sum_{k=0}^{\infty} (f, \phi_k)(\psi_k, g),$$

for all $f, g \in \mathcal{D}$

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